Indian Statistical Institute, Bangalore
M. Math. Second Year, Second Semester
Advanced Functional Analysis

Back paper Examination Date: 28-05-2015
Maximum marks: 100 Time: 3 hours

(1) Let X be a topological vector space. A subset A of X is said to be bounded if for every neighborhood V of 0, there exists $s > 0$, such that $A \subset tV$ for all $t > s$. Suppose A, B are bounded subsets of X, show that $A \bigcup B$ and $A + B$ are bounded. [15]

(2) Let Y be a normed linear space. Let D be a non-empty subset of Y. Show that D is bounded if and only if for every $f \in Y^*$ (Y^* is the space of bounded linear functionals on Y),
$$\sup\{|f(a)| : a \in D\} < \infty.$$
[15]

(3) Let $\mathcal{H}$ be a Hilbert space with ortho-normal basis $\{e_n : n \in \mathbb{N}\}$. Let $M =$ span $\{e_n : n \in \mathbb{N}\}$. Consider the set $S = \{\frac{1}{n}e_n\} \bigcup \{0\}$. Show that S is compact. Show that the closed convex hull of S is not compact in M, but it is compact in $\mathcal{H}$. [20]

(4) Let A be a bounded operator on a Hilbert space. Show that
$$\|A\|^2 = \|A^*A\|.$$
[15]

(5) Let $\mathcal{A}$ be a unital C^* algebra. Show that a linear functional ϕ on $\mathcal{A}$ is positive if and only if $\|\phi\| = \phi(1)$. [15]

(6) Let $\mathcal{C}$ be the C^* algebra of 2×2 matrices. For a matrix $X = [x_{ij}]$, define $\phi(X) = \frac{1}{3}(x_{11} + 2x_{22})$. Show that ϕ is a state on $\mathcal{C}$. Describe the GNS triple $(\mathcal{H}, \pi, \xi)$ of ϕ. Compute the dimension of $\mathcal{H}$. [20]

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